

## Algebra Qualifying Exam

Fall 2014

You have 3 hours to answer all questions.

1. Determine the number of conjugacy classes of elements of order 4 in  $GL_4(\mathbb{C})$ , and give a representative of each class. Do the same for  $GL_4(\mathbb{F}_7)$ .

2. Find the Galois groups of  $f(x) = x^5 + 7x^3 + 6x^2 + x + 5$  over  $\mathbb{F}_2$ ,  $\mathbb{F}_3$ ,  $\mathbb{F}_5$ , and  $\mathbb{Q}$ . You may assume without proof that  $f(x) \in \mathbb{F}_3[x]$  has no irreducible quadratic factors.

3. Let  $R$  be a Noetherian ring. For any ideal  $J \subseteq R$  define

$$P_J = \{f \in R : f^k \in J \text{ for some } k \in \mathbb{Z}^+\}$$

If  $P_J = J$ , show that  $J$  can be expressed as a finite intersection of prime ideals. Hint: among all counterexamples, a maximal one cannot be prime.

4. Classify the groups of order  $2915 = 5 \cdot 11 \cdot 53$ .

5. Suppose  $R$  is a PID and  $A$  and  $B$  are  $R$ -modules. Let  $B_{\text{tors}} \subseteq B$  be the submodule of  $R$ -torsion elements. Prove that

$$\text{Tor}_1^R(A; B) = \text{Tor}_1^R(A; B_{\text{tors}}):$$

6. Suppose  $R$  is a Noetherian ring and  $\mathfrak{p} \subseteq R$  is a prime ideal. Show that there is an  $r \notin \mathfrak{p}$  such that  $S^{-1}R \rightarrow R_{\mathfrak{p}}$  is injective, where  $S = \{1, r, r^2, r^3, \dots\}$ .

7. Suppose  $R$  is a commutative local ring, and  $M$  and  $N$  are  $R$ -modules satisfying

$$M \otimes_R N = 0:$$

(a) If  $M$  and  $N$  are finitely generated, show that either  $M = 0$  or  $N = 0$ .

(b) Show by example that (a) is false if we drop the hypothesis that  $M$  and  $N$  are finitely generated.

8. Let  $\zeta_8 \in \mathbb{C}$  be a primitive eighth root of unity. The ring of integers in  $\mathbb{Q}[\zeta_8]$  is  $\mathbb{Z}[\zeta_8]$  (you may assume this without proof). If  $p$  is a prime, determine the number of primes of  $\mathbb{Z}[\zeta_8]$  above  $p$  when

(a)  $p = 2$ ,

(b)  $p \equiv 1 \pmod{8}$ ,

(c)  $p \equiv 3, 5, 7 \pmod{8}$ .