Complex Analysis Qualifying Exam, Spring 2021

This part of the Analysis Qualifying exam has four problems, each worth 10 points. Show all your work and explain all your reasoning. You may use any result proved in our course, as long as you state clearly what result you are using (including its hypotheses). However, you may not use a result which is the same as the problem you are being asked to do.

1. Suppose f is an entire function and there is a polynomial p such that $|f(z)| \leq |p(z)|$ for all $z \in \mathbb{C}$. Prove that f itself is a polynomial of degree at most the degree of p.

2. Find all possible values of

$$\int_\gamma \frac{e^z}{z^2+1}\,dz$$

where γ is a closed path in $\mathbb{C} \setminus \{i, -i\}$.

3. Suppose $\{f_n\}$ is a sequence of nowhere-zero entire functions converging to a function $f : \mathbb{C} \to \mathbb{C}$, and the convergence is uniform on $\{z \in \mathbb{C} : |z| \leq r\}$ for every $r > 0$. Prove that f is either nowhere-zero or identically zero.

4. Let U be a simply connected open subset of $\mathbb{C}$ such that $U \neq \mathbb{C}$ and let $f : U \to U$ be a holomorphic function. Suppose there are two distinct points $a, b \in U$ such that $f(a) = a$ and $f(b) = b$. Prove that $f(z) = z$ for all $z \in U$.